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METHOD FOR SOLVING THE LINEAR MULTICRITERIA OPTIMIZATION PROBLEM IN INTEGERS

Abstract. *A wide range of practical optimization problems in various fields lead to the solution of multicriteria linear optimization models in integers. There is a growing increase in their importance [2]. Into the current paper we propose a method for solving the multicriteria model of linear type in integers of interactive type. Thus, the decision maker, initially assigning a certain utility to each criterion, will finally build a uni- criterion model of linear optimization in integers. The imposition of each criterion quantified in the synthesis function remains at the discretion of the decision maker, the optimal values and weight being calculated in whole or real numbers, which does not change the optimal solution of the model. To this end, the decision-maker has at his disposal a selection of combinatorial values of the objective functions, which depends on the number of criteria in the initial model. When changing the value of utilities, the decision maker can determine a new optimal compromise solution of the initial model. The theoretical justification of the algorithm is brought in the paper. The algorithm was tested on several examples, which proved its veracity.*

Keywords: *Multi-criteria model in integers, efficient solution, optimal compromise solution.*

JEL Classifications: C02, C44, C61

1. Introduction

Although the support for theoretical justifications for solving linear optimization problems in integers is not so consistent and well argued, interest in this type of problem is constantly growing. The major importance of using mathematical optimization in integers is due to the need to obtain integer solutions in various modeled practical situations. This additional condition has a very high cost, as considerable efforts are made to solve such problems. Among the practical fields of application of the solution of the optimization model in integers, a special place belongs to the problem of one, two and three-dimensional cutting [1], [3], [5]. A number of studies can be listed here, such as: dynamic memory allocation,

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solving problems on multiprocessor systems and general positioning problems (Coffman et al. 1978, Garey and Johnson 1981, Coffman and Leighton 1989, Dyckhoff 1990). The two-dimensional variant of the cutting problem is of NP complexity due to its to its combinatorial explosion with increasing size of the problem (Garey and Johnson1979). Several researchers have written various articles and manuscripts on the subject (Dowsland and Dowsland 1992, Sweeny and Paternoster 1992, Dyckhoff 1990, Coffman 1984, Golden 1976, Gilmore 1966). All approaches of these researchers can be divided into 3 categories: accurate, heuristic, metaheuristic. The exact methods were investigated by Gilmore and Gomory (1961) and are considered the first methods actually applied in the tailoring industry. Recently, Cung and other researchers (2000) developed an algorithm, which allows the exact solution of some variants of two-dimensional cutting problems. The main disadvantage of these methods is that they cannot provide good results for large problems. But when the problem is of multicriteria type, even linear, this effort is further amplified. That said, the condition that the decision variables belong to the set of integers creates a major difficulty, the problem gaining another level of complexity and is solved in a longer time [4], [12, [13], [14], [15]. Despite the fact that the convergence of the Gomory algorithm [13] applied in solving linear programming problems in integers with a single criterion has not been demonstrated, no examples have been provided that would compromise the method. Thus, applying the given algorithm, we can determine the solution of some problems whose difficulty exceeds the average level in a finite number of iterations. Therefore, the scientific research study for this field remains open [6], [8], [10], given that there is a wide range of multicriteria models of fractional linear type, fuzzy, etc., which for application reasons must be to be solved in whole numbers.

2. Problem formulation

The integer multicriteria linear optimization problem is usually described by a set of linear constraints, such as equations and / or inequalities, including on the variables constraints of non-negativity and integrity. The decisional problem with an infinite number of variants is described as follows:

$$\begin{cases} optimum\{F_k(x)\}, k = \overline{1,r} \\ x \in D \\ D = the\ field\ of\ admissible\ solutions \end{cases} \quad (1)$$

in which: $D = \{x = (x_1, x_2, \dots, x_n)^T \mid Ax \le b, x \in Z^+\}$.

In explicit form the model (1) can be formulated as follows:

$$\begin{cases} \begin{Bmatrix} \min \\ \max \end{Bmatrix} F_k(x) = \sum_{j=1}^{n} c_{kj} x_j, \quad k = \overline{1, r} \\ A \cdot x \le b \\ x \in Z^+ \end{cases} \tag{2}$$

where : $A = \|a_{ij}\|$ is an array of size $m \times n \ (m < n)$, $C = \|c_{kj}\|$, is an array of size $r \times n \ (r < n)$, x is a vector n-dimensional column, and b is a m-dimensional column vector.

The interpretations of the parameters c_{kj} may be the most different, according of their practical meanings such as unit costs or benefits, or others close in meaning. Their significance determines the type of the corresponding objective function, minimum or maximum. Analogously, the elements of the matrix A, a_{ij}, represent the specific consumption of the resource j for the production of a product unit of type i, and the elements of the vector b represent the available by types of resources.

We note that in model (2) it is possible to have some criteria of minimum type and others of maximum type, for example, maximizing benefits, profit or others or minimizing costs, depreciation, loss or others.

3. Theoretical landmarks

In order to solve the multicriteria optimization model in integers [11], we will propose some analogous approaches to those in real numbers.

1. The solution $x^* \in Z^+$ is the vector that optimizes a synthesis function of r objective functions, i.e.: $h(F) = h[F_1, F_2, \ldots, F_r]$, in which $h(.)$ it can be defined in several, various ways.
2. The solution $x^* \in Z^+$ is the vector which minimizes one criterion in the form of: $$\phi(x^*) = \min_{x \in D} h(\psi_1(x - X_1), \ldots, \psi_r(x - X_r)),$$

in which, $X_j = (x_{1j}, x_{2j}, \ldots, x_{nj})^T$, $j = \overline{1, r}$ is the optimal solution to the problem with a single objective function, F_j, and ψ_k is a distance type function between the vector $x \in D$ and optimal solution X_k for the corresponding criterion F_k.

4. The solution $x^* \in Z^+$ is the vector which belongs to a set of effective whole-type points.

Because the model (2) is of multi-criteria type, it's known that such kind of model rarely admits the optimal solutions in integers.

Definition 1 *The basic solution* X^* *of the* model (2), *where* $X^* \in Z^+$ *is called optimal overall if it is the optimal solution for each of criteria.*

By solving model (2) we will assume the construction of a finite set of its efficient integer solutions known again as a Pareto-optimal or non-dominated solutions [11], solutions of the best compromise. We will further propose the definition of the efficient solution for the multicriteria linear deterministic problem in integers.

Definition 2 *The basic solution* $\overline{X}$, *where* $\overline{X} \in Z^+$ *of the* model (2) *is an basic efficient one if and only if it doesn't exists any other basic solution* $X \in Z^+$, *where* $X \neq \overline{X}$, *which would improve the values of all criteria and at least one criterion would be strictly improved.*

We propose the same definition in a more rigorous form.

Definition 3 *The basic solution* $\overline{X} \in Z^+$ *of the* model (2) *is a basic efficient one if and only if or any other basic solution* $X \in Z^+$, *where* $X \neq \overline{X}$, *for which are true the relation:* $F_{j_1}(X) \geq F_{j_1}(\overline{X})$, *where* $j_1 \in (1,...j_2)$, *indices corresponding to the* maximum *type of criteria immediately exists at least one index* $\exists j_l \in (j_2+1,...,r)$, *indices of the* minimum *type of criteria for which the relation:* $F_{jl}(X) > F_{j_1}(\overline{X})$ *is true or, if the relation:* $F_{jl}(X) \leq F_{j_1}(\overline{X})$ *is true for all indices corresponding to the* minimum *type of criteria which are* $j_l \in (j_2+1,...,r)$, *imediately exists at least one index* $\exists j_1 \in (1,...j_2)$ *from the set of indicis of the* maximum *criteria type, for which the relation* $Z_{j_2}(\overline{X}) < Z_{j_2}(X)$ *is true.*

Definition 4 *The basic solution* X^* *of the* model (2), *where* $X^* \in Z^+$ *is one of the optimal (best) compromise solution if it is located closed to the optimal solutions of each criterion.*

4. Section plan methods

The section plan methods perform a procedure for iteratively improving the solution of the optimization model in integers by sectioning the range [7] of admissible solutions using various section plans, built according to certain rules. The algorithm of section plans is also called "Cyclic Algorithm". Iteratively, the components of the optimal solution are modified, its belonging to the admissible domain of the model as well as the evolution of the value of the objective function in the sense of optimizing the unicriteria linear model.

The algorithm is convergent and finite. Although its convergence has not been demonstrated, no example has been found that would contradict the algorithm. Therefore, after a finite number of steps, the algorithm determines the optimal solution of the model, which is of type integer, of course, if it exists. The proposed algorithm was developed by American scientist R. Gomory and was first published by him in 1958. For these reasons, the name "Gomory Algorithm" [13] is often used in the literature, instead of the method of section.

We will consider the following couple of issues:

$$\text{(ILP)} \begin{cases} (\max) f = c \cdot x \\ A \cdot x = b \\ x \geq 0 \\ x \in Z^+ \end{cases} \qquad \text{(LP)} \begin{cases} (\max) f = c \cdot x \\ A \cdot x = b \\ x \geq 0 \end{cases}$$

where the elements of the matrix *A* and the components of the vector *b* are of integer type. We noted : $D_0 = \{x / A \cdot x = b,\ x \in Z^+\}$, *and* $D = \{x / A \cdot x = b,\ x \geq 0\}$, therefore D_0 is the domain of admissible solutions to the problem *(ILP)* and D that of the problem (LP), respectively.

Stages of realization and justification of the plan section algorithm

We will initially assume that x^* it does not have all the integer components. In this case an unverified constraint of the fractional optimum x^*, but satisfied by any admissible solution of whole type is constructed. This restriction is added to the original problem noted with (LP_0), after which the optimal solution will be re-optimized. Let x^{**} be the optimal solution to the augmented problem, denoted by *(*LP_1*)*. Due to the way in which the additional restriction was defined, we have the following true relationships between the admissible domains:

$$D_{ILP} \subset D_{LP_1} \subset D_{LP_0} = D_{LP}.$$

If x^{**} it does not have all the components integers, the procedure is repeated: a new constraint is built unverified by x^{**}, but verified by the whole admissible solutions. The new constraint is added to *(*LP_1*)*, resulting in a linear programming problem *(*LP_2*)*. The construction procedure is the next:

$$D_{ILP} \subset D_{LP_2} \subset D_{LP_1} \subset D_{LP_0} = D_{LP}.$$

After applying the re-optimization procedure, it is decided whether LP_2 admit or not optimal solution. The theory assures us that, under certain conditions, after a finite number of steps, we reach a linear programming problem, (LP_{k-1}), whose optimal solution is $x^{k(*)}$, which has all its components of integer type, so it is the optimal solution of the problem (ILP).

From a geometric point of view, each new constraint removes a certain portion of the set of admissible solutions to the previous problem, making a section of the admissible domain and cutting the intrusive section, hence the name of the cut given to these additional constraints. From a mathematical point of view, the sectioning algorithm [13] is made as follows.

We will consider the matrix $\overline{A}$, the vector $\overline{b}$ are the ones corresponding to the optimal solution of the model *(LP).* We will assume that not all components of vector $\overline{b}$ are of type integer. Let be the component r of the vector $\overline{b}$ with the maximum fractional part, this being $\overline{b}_r$.

We will apply the following r representation to the constraint coefficients:

$$\overline{b_r} = x_r + \sum_{j \in J} \overline{a_{rj}} x \tag{3}$$

which can be decomposed in this way**:**

$$\left\lfloor \overline{b_r} \right\rfloor + \left\{ \overline{b_r} \right\} = x_r + \sum_{j \in J} \left(\left\lfloor \overline{a_{rj}} \right\rfloor + \left\{ \overline{a_{rj}} \right\} \right) \cdot x_j \tag{4}$$

The hypothesis admitted on $\overline{b_r}$ implies the following relation: $0 < \left\{ \overline{b_r} \right\} < 1$.

Therefore, the following equality is true:

$$\left\lfloor \overline{b_r} \right\rfloor - \sum_{j \in J} \left\lfloor \overline{a_{rj}} \right\rfloor \cdot x_j - x_r = \sum_{j \in J} \left\{ \overline{a_{rj}} \right\} \cdot x_j - \left\{ \overline{b_r} \right\} \tag{5}$$

Let x a whole admissible solution to the problem (ILP). In this case, the left member of the relation (5), is an integer, therefore we get true the next relation:

$$\left\lfloor \overline{b_r} \right\rfloor - \sum_{j \in J} \left\lfloor \overline{a_{rj}} \right\rfloor \cdot x_j - x_r \in Z \tag{6}$$

So, the right member of equality (5) calculated in the same solution is an integer, i.e. we have true the relation:

$$\sum_{j\in J}\left\{\overline{a_{rj}}\right\}\cdot x_j - \left\{\overline{b_r}\right\} \in Z \qquad (7)$$

We will prove that the relationship is true:

$$\sum_{j\in J}\left\{\overline{a_{rj}}\right\}\cdot x_j - \left\{\overline{b_r}\right\} \geq 0 \qquad (8)$$

Proof. We will assume, by absurdity, the opposite, that is:

$$\sum_{j\in J}\left\{\overline{a_{rj}}\right\}\cdot x_j - \left\{\overline{b_r}\right\} < 0 \qquad (9)$$

Then from the relation (5) results the inequality:

$$\left\lfloor\overline{b_r}\right\rfloor - \sum_{j\in J}\left\lfloor\overline{a_{rj}}\right\rfloor\cdot x_j - x_r < 0 \qquad (10)$$

and from (6) it follows that the next true expression:

$$\left\lfloor\overline{b_r}\right\rfloor - \sum_{j\in J}\left\lfloor\overline{a_{rj}}\right\rfloor\cdot x_j - x_r \leq -1 \qquad (11)$$

From the equality (5) we will deduce the following relations:

$$\sum_{j\in J}\left\{\overline{a_{rj}}\right\}\cdot x_j - \left\{\overline{b_r}\right\} \leq -1 \text{ adică: } \sum_{j\in J}\left\{\overline{a_{rj}}\right\}\cdot x_j \leq \left\{\overline{b_r}\right\} - 1 \qquad (12)$$

Because we have $\left\{\overline{a_{rj}}\right\} \geq 0,\ (\forall)\, j\in J$, we get that left member of the relationship (5) is also ≥ 0, while the right limb is < 0, because we have $\left\{\overline{b_i}\right\} < 1$. The obtained contradiction demonstrates inequality (8).

Since *x* was chosen arbitrarily, it follows that the following constraint is true:

$$\sum_{j\in J}\left\{\overline{a_{rj}}\right\}\cdot x_j \geq \left\{\overline{b_r}\right\} \qquad (13)$$

and that it is verified by any permissible integer solution.

Otherwise, in the optimal solution, which is not of the whole type x^*, we will have $x_j^* = 0,\ j\in J$, values that are entered in (8) lead to inequality $\left\{\overline{b_r}\right\} \leq 0$, fact, which contradicts hypothesis (4), according to which we have $\left\{\overline{b_r}\right\} > 0$. Consequently, the inequality (10) is not verified by the fractional optimum x^*. We will add the constraint (10) to the problem (LP), and get the modified problem with *(m + 1)* constraints, which we will denote by (LP_1).

In order to apply the solution re-optimization procedure, we will initially transform the restriction (10) into equality, introducing a deviation variable x_{n+1}. Thus, we will introduce a new restriction: $-\sum_{j\in J}\left\{\overline{a_{rj}}\right\}\cdot x_j + x_{n+1} = -\left\{\overline{b_r}\right\}$, which is considered a *sectioning restriction.*

5. Method of maximizing global utility

The method of maximizing global utility was developed by Gh., Boldur Lăţescu and I.M. Stancu - Minasian in [11]. It is based on the idea of transforming the objective functions of a multicriteria problem into utility functions in the sense of von Neumann - Morgenstern, which are to be summed to obtain a synthesis function.

This method, developed in the hypothesis of the existence of a linear programming problem with multiple criteria, can be used quite efficiently even in the case of decision problems in which we have an infinite number of variants.

Definition 5 Given n criteria: $C_1, C_2, \ldots, C_n$, *they are called mutually independent in the sense of the theory of utility, if and only if we have the true relation:* $\omega_1 \sim \omega_1$ *for anything* $(\omega_1, \omega_2) \in G$.

In this case the additives of the utilities is possible and, obviously, we will have true the next relation:

$$U(a_{i1}, a_{i2}, \ldots a_{in}) = u_1(a_{i1}) + u_2(a_{i2}) + \cdots + u_n(a_{in}).$$

Intuitively, the independence of the criteria in the sense of utility theory specifies that a consequence of any variant V_i, from the point of view of the criterion C_k, always corresponds to the same utility, no matter what consequence, from the point of view of this criterion, is associated.

For the presentation of the method we will resume the next linear multicriteria decision-making problem (1):

$$\begin{cases} optim\{F_j(x)\}, j = \overline{1,r} \\ x \in D \end{cases} \tag{14}$$

where: F_j are the multiobjective (linear) functions / criteria, and *D* is the range of admissible solutions defined by a set of linear constraints, including the positive conditions of the variables:

$$D = \{x \in R \mid Ax \le b, x \ge 0\}.$$

Global utility maximization method algorithm [11]:

Step 1. For each function - purpose its optimal value X_j is determined, where $F_j = \underset{x \in D}{optim} F_j(x)$ and Y_j is its pessimistic value, where $F_j^p = \underset{x \in D}{pessim} F_j(x)$.

Step 2. For the set of optimal and pessimistic sets of values of all criteria, the corresponding value utilities in the sense Neumann - Morgenstern [9] are associated as follows:

$$\{F_1, F_2, \ldots, F_r; F_1^p, F_2^p, \ldots, F_r^p\} \to \{U_1, U_2, \ldots, U_r; U_{r+1}, U_{r+2}, \ldots, U_{2r}\}.$$

Step 3. The objective functions F_j are transformed into utility functions of the type FU_j, initially solving *r* linear systems with *2r* variables, the unknowns being the coefficients of the type: $\{(\alpha_j, \beta_j)\}_{j=\overline{1,r}}$.

After solving the *r* systems of linear equations of this type:

$$\begin{cases} \alpha_j F_j + \beta_j = u_j \\ \alpha_j F_j^p + \beta_j = u_{j+r} \end{cases}, \quad j = \overline{1,r}$$

we will construct accordingly *r* utility functions such as:

$$FU_j = \alpha_j F_j(X) + \beta_j, \; j = \overline{1,r}$$

Step 4. Finally, we will solve the problem of linear programming aimed at maximizing the global utility - UG, which is as follows:

$$\max_{x \in D} UG = \max_{x \in D} \sum_{j=1}^{r} \pi_j FU_j \;,$$

where π_j - is the importance coefficient of the criterion C_j, which, obviously, can be changed by the decision maker, thus obtaining a new optimization problem.

6. Combinatorial synthesis algorithm for solving the linear multicriteria optimization

One of the most important problems that arises when solving the multicriteria optimization problem in integers using the methods of synthesis functions is: what kind of optimal solutions of each criterion we will use to build the synthesis function of all criteria, these being in R^+ or in Z^+, so that the final model solve it in Z^+? In this justified paragraph we will answer this question.

In order to solve the multicriteria model of linear optimization in integers of type (2) we will apply the method of synthesis functions, namely we will use the method of maximizing global utility, which we will achieve in two stages.

Stage I

1. At this stage we will solve *2r* unicriteria linear programming problem from model (2) of type: $F_j = \underset{x \in D}{optim} F_j(x)$ and $F_j^p = \underset{x \in D}{pessim} F_j(x)$, on the admissible domain: $D = \{x \in R \mid Ax \le b, x \ge 0\}$;
2. Next we will solve *2r* more linear programming problems of the type: $F_j = \underset{x \in D}{optim} F_j(x)$ and $F_j^p = \underset{x \in D}{pessim} F_j(x)$, on the admissible domain:
$$D = \{x \in Z^+ \mid Ax \le b, x \ge 0\};$$
3. We will combinatorial select the vectors of optimal values and corresponding to the pessimistic values of the objective functions, some calculated on Z^+, others in R^+. The number of such combinations is finite because the size of the problem is finite. These can be described as follows:

$$\left\{ \begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(R^+) \end{pmatrix} \vee \begin{pmatrix} F_1(R^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \vee \begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \vee \dots \vee \begin{pmatrix} F_1(Z^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(R^+) \end{pmatrix} \vee \begin{pmatrix} F_1^p(R^+) \\ F_2^p(Z^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix} \vee \begin{pmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix} \vee \dots \vee \begin{pmatrix} F_1^p(Z^+) \\ F_2^p(Z^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix} \right\}$$

The number of such vectors is: $N(V) = C_r^1 + C_r^2 + \dots + C_r^r$, the same as the number of vectors with pessimistic value records of the criteria.

Stage II

1. By selecting one of the vector records of the values of the objective functions and the vector of the corresponding records of the pessim values, we will construct the synthesis function, which expresses the summary utility of the criteria: $G = \sum_{j=1}^{r} (\alpha_j F_j + \beta_j)$, which must be maximized. The coefficients $\{(\alpha_j, \beta_j)\}_{j=\overline{1,r}}$ are determined by applying the global utility maximization algorithm, described above.

2. We will determine the optimal solution of the next model: $\max_{x \in D} G = \sum_{j=1}^{r} \left(\alpha_j F_j(X) + \beta_j\right)$, where: $D = \{x / A \cdot x = b,\ x \in Z^+\}$, that is the optimal compromise solution for model (2). Either that is it X^*. We will calculate the values of each objective function in this solution and we will build the next vector of records of the criteria: $\begin{Bmatrix} F_1(X^*) \\ F_2(X^*) \\ \ldots \\ F_r(X^*) \end{Bmatrix}$.

Theorem For a set of a priori utilities assigned to the criteria in model (2), the solution of the optimal compromise of the integer model remains the same for any vector of the optimal records of the combinatorial criteria calculated in R^+ or in Z^+.

Proof. Let X_{eff}^1 be a solution of the optimal compromise for the whole type model (2), which records the smallest distance to the optimal whole type solutions of each criterion. We will assume that the synthesis function of the final solved model was constructed using a combination of optimal values of the objective functions from model (2), some being solved in R^+, others in Z^+.

Let: $\begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \ldots \\ F_r(Z^+) \end{pmatrix}$ -vector of the optimal and pessim $\begin{pmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \ldots \\ F_r^p(R^+) \end{pmatrix}$ recorded values of objective functions.

We will assume that for another recording values of the objective functions from the model (2), different from the previous one, let it be: $\begin{pmatrix} F_1(Z^+) \\ F_2(R^+) \\ \ldots \\ F_r(Z^+) \end{pmatrix}$, and corresponding vector of the pessim values $\begin{pmatrix} F_1^p(Z^+) \\ F_2^p(R^+) \\ \ldots \\ F_r^p(Z^+) \end{pmatrix}$, the objective synthesis function registered another solution of the optimal compromise in integers, different from the first, either it is X_{eff}^2. If $X_{eff}^1 \neq X_{eff}^2$, then there is at least one coordinate after which these vectors differ. Therefore, at least for one criterion, let it be with indexes i_1, the distance between its optimal solution in integers and the new solution is smaller than the previous one, i.e. the relationship is fair: $\rho\left(X_{eff}^{1}, X_{i_1}^{*}\right) > \rho\left(X_{eff}^{2}, X_{i_1}^{*}\right)$, where $X_{i_1}^{*}$ is optimal solution in integer of criterion i_1, which contradicts the assumption that X_{eff}^{1} is the solution of the optimal compromise in integers for the model (2), which had to be demonstrated. So, our assumption is wrong. Therefore, model (2) admits a single solution of the optimal compromise in integers, regardless of the configuration of records of the optimal values of the criteria in R^{+} or Z^{+}, used in the construction of the synthesis function of the model.

Remark 1. For any vector of combinatorial records of the values of the objective functions of the unicriteria models of the problem (2) in R^{+} or in Z^{+}, and for their a priori known utilities, the optimal compromise solution of the model (2) in integers remains the same.

Remark 2. For any new set of initial utilities assigned to the criteria in model (2), applying the method of maximizing the maximum utility we will obtain a new solution of the optimal compromise in integers for all the criteria of this model.

7. Conclusions

It is well known that multicriteria optimization models enjoy a growing interest in everyday life, especially because they more appropriately describe decision-making situations in different socio-economic fields. Unfortunately, we do not always have methods to solve them efficiently. Imposing the solution in integers of the model, of course, increases the complexity of the problem, even in the case of the linear type model. The proposed paper brings a clarity in the problem of solving the multicriteria optimization model of linear type in integers. We focused on the use of the methods of synthesis functions, namely the method of maximizing the global utility in solving the multicriteria model of linear type in integers, which leads us to determine an optimal compromise solution, closest to the optimal solutions in integers of each separate criterion. To determine this, the decision maker can use both the optimal value of each criterion in integers and the one calculated on the set of real numbers, of course positive. The set of all possible combinations of such vectors for recording the values of the objective functions as well as the weight values was exploited. Regardless of the configuration used to construct the synthesis function, its optimal solution in integers does not change. So, the decision maker can select the most advantageous values in terms of calculations of objective functions, a very important moment that certainly increases the efficiency of the algorithm. This conclusion was theoretically justified by the theorem that is brought in the paper. The proposed algorithm was verified on a concrete example proposed in the paper.

Example

For the following linear model of multicriteria optimization in integers find the solution of the optimal compromise using the method of synthesis functions, for the proposed utilities of criteria.

$$\min\{F_1(X) = x_1 + 2x_2 + x_3\}$$
$$\max\{F_1(X) = 2x_1 + x_2 + 2x_3\}$$
$$\max\{F_1(X) = 2x_1 + 3x_2 + x_3\}$$
$$\begin{cases} 3x_1 + 5x_2 + x_3 \leq 18 \\ 5x_1 + 3x_2 + 2x_3 \leq 20 \\ 2x_1 + x_2 + 2x_3 \geq 5 \\ x_j \in Z^+ \end{cases}$$

F_1	F_2	F_3	F_1^p	F_2^p	F_3^p
U_1=4	$U_2 = 8$	$U_3 = 9$	$U_1 = 1$	$U_2 = 2$	$U_3 = 2$

Solving procedure: In order to solve the problem, among the methods of synthesis functions we will apply the method of maximizing the global utility. We will initially solve six unicriteria linear programming problems in R^+, determining the optimal and pessim values of each criterion according to Stage I of the proposed algorithm. Next, we will solve the same six unicriteria linear programming problems in Z^+, similarly determining the optimal and pessim values of each criterion. For the construction of the synthesis function, using the method of maximizing the global utility, we found the combinatorial form of the registration vectors of the optimal and pessim values of the objective functions in R^+ and Z^+. These are the following:

1) $\begin{Bmatrix} F_1(R^+) \\ F_2(R^+) \\ F_3(R^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ F_3^p(R^+) \end{Bmatrix}$; 2) $\begin{Bmatrix} F_1(Z^+) \\ F_2(Z^+) \\ F_3(Z^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(Z^+) \\ F_2^p(Z^+) \\ F_3^p(Z^+) \end{Bmatrix}$; 3) $\begin{Bmatrix} F_1(R^+) \\ F_2(Z^+) \\ F_3(Z^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(R^+) \\ F_2^p(Z^+) \\ F_3^p(Z^+) \end{Bmatrix}$;

4) $\begin{Bmatrix} F_1(R^+) \\ F_2(R^+) \\ F_3(Z^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ F_3^p(Z^+) \end{Bmatrix}$; 5) $\begin{Bmatrix} F_1(R^+) \\ F_2(R^+) \\ F_3(R^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ F_3^p(R^+) \end{Bmatrix}$; 6) $\begin{Bmatrix} F_1(Z^+) \\ F_2(Z^+) \\ F_3(R^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(Z^+) \\ F_2^p(Z^+) \\ F_3^p(R^+) \end{Bmatrix}$;

7) $\begin{Bmatrix} F_1(Z^+) \\ F_2(R^+) \\ F_3(Z^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(Z^+) \\ F_2^p(R^+) \\ F_3^p(Z^+) \end{Bmatrix}$; 8) $\begin{Bmatrix} F_1(Z^+) \\ F_2(R^+) \\ F_3(R^+) \end{Bmatrix}, \begin{Bmatrix} F_1^p(Z^+) \\ F_2^p(R^+) \\ F_3^p(R^+) \end{Bmatrix}$;

We will not place the optimal solutions and the weight of the solved models, but we will place them directly in the vectors of the proposed combinations of values. Next we will solve 24 systems of linear equations in order to determine the coefficients $\{(\alpha_j, \beta_j)\}_{j=\overline{1,r}}$ needed to construct the function summary utility function. Analogously, we will not place the solutions of the systems of solved equations, but we will directly propose the objective functions, which represent synthesis functions for each of the eight listed cases. For each of the proposed combinations we will describe the corresponding synthesis functions constructed using the same table of proposed utilities for the model criteria. These are the next:

$$F_1(U) = 1.73x_1 + 1.63x_2 + 1.09x_3 \to \max$$
$$F_2(U) = 1.83x_1 + 1.75x_2 + 1.13x_3 \to \max$$
$$F_3(U) = 1.85x_1 + 1.8x_2 + 1.15x_3 \to \max$$
$$F_4(U) = 1.85x_1 + 1.8x_2 + 1.15x_3 \to \max$$
$$F_5(U) = 1.73x_1 + 1.63x_2 + 1.09x_3 \to \max$$
$$F_6(U) = 1.7x_1 + 1.57x_2 + 1.07x_3 \to \max$$
$$F_7(U) = 1.83x_1 + 1.75x_2 + 1.13x_3 \to \max$$
$$F_8(U) = 1.7x_1 + 1.57x_2 + 1.07x_3 \to \max$$

They express the summary utility of all the criteria and are to be maximized on the admissible domain of values, which is given by the same restrictions:

$$\begin{cases} 3x_1 + 5x_2 + x_3 \le 18 \\ 5x_1 + 3x_2 + 2x_3 \le 20 \\ 2x_1 + x_2 + 2x_3 \ge 5 \\ x_j \in Z^+ \end{cases}$$

Solving in turn these 8 problems of linear programming in integers, we obtained the following solutions of the optimal compromise:

$$X_{eff}^1 = X_{eff}^2 = X_{eff}^3 = X_{eff}^4 = X_{eff}^5 = X_{eff}^6 = X_{eff}^7 = X_{eff}^8 = \{x_1 = 1, x_2 = 3, x_3 = 0\};$$

We calculated the values of the utility functions, which are the following:

$F_1(U) \approx 11.89$; $F_2(U) \approx 12.105$; $F_3(U) \approx 12.02$; $F_4(U) \approx 12.02$;
$F_5(U) \approx 11.89$; $F_6(U) \approx 11.93$; $F_7(U) \approx 12.105$; $F_8(U) \approx 11.93$;

$$F(X_{eff}^1) = \begin{Bmatrix} F_1(X_{eff}^1) \\ F_2(X_{eff}^1) \\ F_3(X_{eff}^1) \end{Bmatrix} = F(X_{eff}^2) = \begin{Bmatrix} F_1(X_{eff}^2) \\ F_2(X_{eff}^2) \\ F_3(X_{eff}^2) \end{Bmatrix} = F(X_{eff}^3) = \begin{Bmatrix} F_1(X_{eff}^3) \\ F_2(X_{eff}^3) \\ F_3(X_{eff}^3) \end{Bmatrix} =$$

$$F(X_{eff}^4) = \begin{Bmatrix} F_1(X_{eff}^4) \\ F_2(X_{eff}^4) \\ F_3(X_{eff}^4) \end{Bmatrix} == F(X_{eff}^5) = \begin{Bmatrix} F_1(X_{eff}^5) \\ F_2(X_{eff}^5) \\ F_3(X_{eff}^5) \end{Bmatrix} = F(X_{eff}^6) = \begin{Bmatrix} F_1(X_{eff}^6) \\ F_2(X_{eff}^6) \\ F_3(X_{eff}^6) \end{Bmatrix} =$$

$$= F(X_{eff}^7) = \begin{Bmatrix} F_1(X_{eff}^7) \\ F_2(X_{eff}^7) \\ F_3(X_{eff}^7) \end{Bmatrix} = F(X_{eff}^8) = \begin{Bmatrix} F_1(X_{eff}^8) \\ F_2(X_{eff}^8) \\ F_3(X_{eff}^8) \end{Bmatrix} = = \begin{Bmatrix} 7 \\ 5 \\ 11 \end{Bmatrix};$$

As we see, the decision maker is free to choose the vector of the registered combinations of the values of the objective functions in R^+ and in Z^+, this doesn't influencing the solution of the optimal compromise for the multicriteria problem of linear type in integers. We would like to mention that by modifying the table of utilities associated with the criteria for the reasons of the real decision-making situation, the decision maker can obtain another solution of the optimal compromise in integers for the model (2), using the algorithm proposed in the paper.

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